Growth-direction dependence of steady-state Saffman-Taylor flow in an anisotropic Hele-Shaw cell

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Selection of steady-state fingers has been measured in a Hele-Shaw cell perturbed by having a square lattice etched onto one of the plates. Flows at different orientations θ between the direction of flow and the lattice axes have been studied, in a wide range of observable tip velocities where the perturbation was made microscopic in the sense that the capillary length of the flow was much greater than the etched lattice cell size. The full range of dynamically interesting angles for the square lattice was examined, and above a threshold, the microscopic perturbation always results in wider fingers than are selected in the unperturbed case. There is some dependence of the width of the fingers on the orientation of the flow, with fingers at $\theta=0°$ being the widest with respect to the unperturbed fingers, and fingers at 45° being the least wide, although still wider than the unperturbed fingers. All observed solutions are symmetric, centered in the channel, and have the relation between tip-curvature and finger width expected of members of the Saffman-Taylor family of solutions. Selected solutions narrow again at tip velocities where the perturbation can no longer be considered microscopic. [S1063-651X(96)00408-4]

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I. INTRODUCTION

Saffman-Taylor (ST) flow, the motion of an interface between immiscible fluids in a rectangular Hele-Shaw cell, has proven to be an interesting case of relatively simple, nonlinear pattern formation [1,2]. In such a system a steady-state viscous finger can be formed, whose selection from a family of possible solutions has only recently been understood [3–5], once the importance of the surface tension to the selection was realized [6–9]. There have been a number of recent experiments which investigated the effects of adding perturbations to the problem of Saffman-Taylor viscous fingering for both radial and channel flows [10–23]. Relatively small perturbations to such flows can lead to dramatic changes in the experimentally observed solutions, ranging from straightforward viscous fingering to the evolution of dendritic patterns. The majority of these experiments in the channel geometry have involved macroscopic perturbations in the sense that the perturbations which were added had length scales larger than the capillary lengths of the moving interfaces [10,13,15,17,21–23]. Most of these experiments tended to select fingers which were narrower and more stable than those found in unperturbed ST fingering. The available theoretical work also tends to favor narrow fingers for the generalized case of anisotropy added by an imposed perturbation [24–34]. An interesting exception involves high velocity flow over a lattice, in which the fingers exhibit very unstable diffusion-limited-aggregation (DLA) -like behavior which reflects the symmetry of the underlying lattice [21,23]. Averaging over a large number of flows produces a finger which has the general shape of an ST finger, but which is sometimes wider or narrower than the corresponding ST finger should be, depending on the orientation of the lattice with respect to the flow. Most of the theoretical work done on flow over a lattice in the channel geometry presupposes that the anisotropy will enter as a direction dependent term in the surface tension [23,25,27,39,40]. That is, there will be a maximum and a minimum in the surface tension depending on the direction of flow with respect to the lattice axes. It should be noted, however, that kinetic (velocity dependent) anisotropy terms are also expected to become observable in the channel geometry above some at present unspecifiable velocity, and these have in fact been seen in radial flow over a lattice [18,20,35]. In this paper, we present results of an experimental investigation of steady-state flow in the channel geometry over a lattice whose length scales were less than the capillary length of the flow. These flow realizations were studied at different orientations of the lattice axes (defined to be in the directions of the grooves) to the direction of flow, to investigate the effect of the underlying fourfold symmetry of the square lattice on the steady-state flow. In a previous work [36], we have shown that if the imposed perturbation has a length scale smaller than the capillary length of the flow, fingers were obtained which were wider and less stable than those found in ordinary ST fingering. This effect was strongly dependent on both the symmetry of the lattice (fourfold vs twofold) and the strength of the perturbation (the ratio between the etching depth and the cell gap). In that work, we also tried to investigate the effect of the orientation of the direction of flow to the lattice axes, although we were limited to angles of $\theta=0°$ to 10° for the square lattice, and angles of $\theta=0°$ to 45° for the rectangular lattice, due to the design of our Hele-Shaw cell. Within these limits we found a weak dependence of finger width and stability on the orientation θ between the lattice symmetry axes and the direction of the flow. The full range of dynamically interesting angles for the rectangular lattice would be $\theta = 0°–90°$ and for the square lattice $\theta=0°–45°$. In our present work, we have made a new Hele-Shaw cell which lets us study fingering at any orientation for the square lattice, allowing a more complete investigation of the possible effects of the angular dependent surface tension term in the boundary conditions.

II. EXPERIMENTAL PROCEDURES AND DATA PRESENTATION

The experimental cell was a modified radial Hele-Shaw cell. The channel width w was set by the parallel placement of Teflon spacers, which also set the gap b between the plates of the cell. In this experiment, the channel width was 3 cm, except for a few cases which will be discussed below, and the gap was $b=0.37$ mm. The top plate of this cell consisted of a 1 in. thick piece of optically flat glass. The bottom plate was a circuit board with a uniform pattern of squares etched into the copper, resting on a bottom glass piece designed to limit flexing in the board. The lattice had a groove width of 200 ± 20 μm, a center to center spacing of 390 ± 20 μm, and an etching depth of 90 ± 10 μm. The channel width and orientation to the lattice were defined by the placement of Teflon spacers of thickness 0.37 mm, and these could be fixed to allow us to sample the complete range of dynamically interesting lattice orientations. Some calibration data were also taken with a smooth bottom plate, to allow us to compare our results to the nonperturbed case. A flow realization consisted of filling the cell with heavy paraffin oil (viscosity $\mu=180$ cP and surface tension T=35.4 dyne/cm), then injecting nitrogen gas into the cell. Starting from an initially flat interface, steady-state fingers were typically formed before the interface had traveled 3 cm and could thus be observed through a flow length of approximately 15 cm in the 25 cm long cell before end effects disturbed the steady state. Only flows with a very constant areal injection rate (variation less than 5% over a 15 cm finger-tip advance) were accepted and analyzed. A CCD camera was used with a medical grade vcr to record the flows, and the flows were then digitized and analyzed using OPTIMAS pattern analysis software [37]. Measurements were made of the tip velocities v, the finger widths λw, and the limits of pattern stability as a function of the angle θ, between the direction of flow and the symmetry axis of the lattice. While the finger width in any one flow realization could be measured with $\pm1.5\%$ accuracy, repeated measurements of fingers at a given velocity showed the finger width to be reproducible only within $\pm5\%$.

III. RESULTS

Below we present finger widths λ (fraction of cell width w, occupied by the fingers) vs the dimensionless capillary number

$$1/B=\frac{12\mu v w^2}{Tb^2}, \tag{1}$$

where μ is the viscosity of the fluid, v is the velocity of the fingertip, w is the width of the channel, and T is the surface tension. We arbitrarily use the gap b as the gap set by the Teflon spacers plus 1/2 the etching depth of the lattice. The range of uncertainty in the gap introduced by the ambiguity in how much of the etching depth to include in $1/B$ introduces an absolute uncertainty in $1/B$ of as much as 20%, but this was small compared to the magnitude of the effects we report below. Only relative uncertainty was included in the error bars, and this was negligibly small in $1/B$. We will concentrate our discussion on the region $100\leq 1/B\leq 3600$

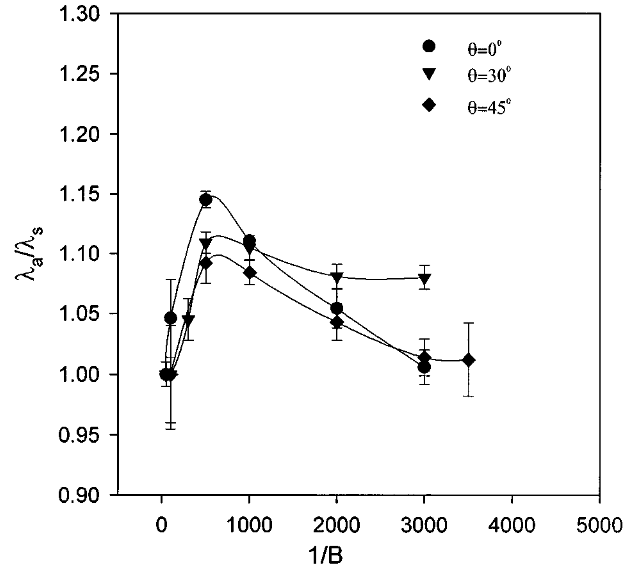

FIG. 1. Dimensionless ratio of perturbed-cell finger width to unperturbed-cell ST finger width for $b=0.42$ mm at all orientations for the square lattice. The lines shown were sketched to guide the eye.

where the lattice perturbation can be considered microscopic, i.e., the region where the capillary length of the fingers l_c was greater than the cell size of the lattice d. The capillary length decreases with increasing velocity and is given by

$$l_c=\frac{Tb}{12\mu v}. \tag{2}$$

At all lattice orientations the fingers displayed several similar behaviors. For very low values of $1/B$, the anisotropy had very little effect, and the finger widths agreed with those found for a smooth cell. As $1/B$ was increased, the fingers widened dramatically off the smooth ST curve. The fingers then remained unusually wide as $1/B$ increased, until the capillary length became of the same order as the cell size, at which point the fingers returned to or near the smooth ST curve. These results hold true in general for fingering over a microscopic lattice with high anisotropy, as we have discussed earlier [36].

Flows at all angles investigated were wider than the smooth ST fingers in the region $100\leq 1/B\leq 3600$, and there was a clear dependence of the width of the finger on the angle of orientation. Flows at 0° showed the widest fingers compared to the unperturbed ST patterns, while flows at 30° were slightly less wide than those seen at 0°, and flows at 45° were the closest to the unperturbed ST results. This behavior can be seen in Fig. 1, where we show the ratio of finger widths λ_a found with the etched cell to finger widths λ_s found with the smooth cell plotted vs $1/B$. Both the numerator and the denominator of each point involves an average finger width λ obtained from several fingers measured at the same value of $1/B$ for either the etched or the smooth cell, and the error bars reflect the 5% fluctuations discussed above for these measured values of λ. In no case were fingers narrower than the unperturbed ST fingers seen in the region

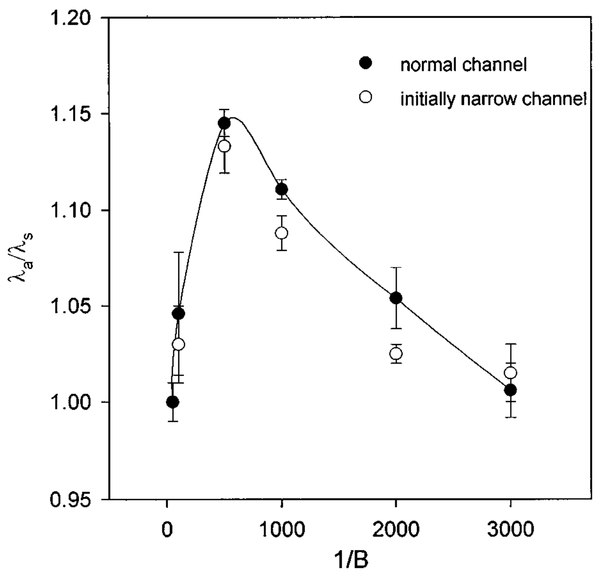

FIG. 2. Dimensionless ratio of perturbed-cell finger width to unperturbed-cell ST finger width for $b=0.42$ mm for flow at 0° over the square lattice. The dark circles represent flow in our usual channel, while the open circles represent flow with an initial narrow finger.

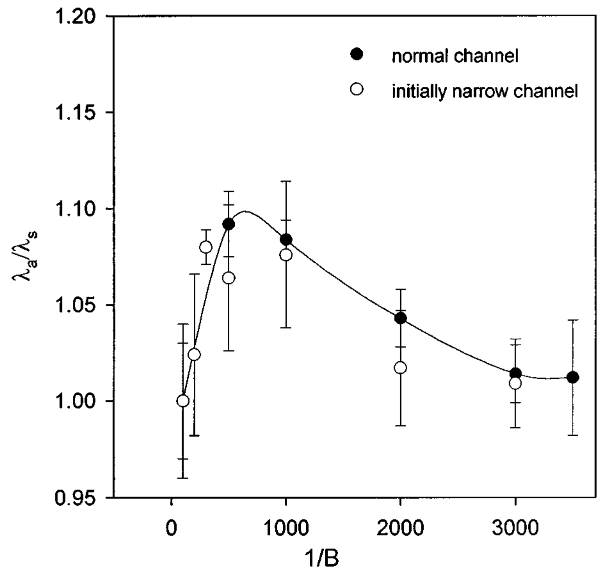

FIG. 3. Dimensionless ratio of perturbed-cell finger width to unperturbed-cell ST finger width for $b=0.42$ mm for flow at 45° over the square lattice. The dark circles represent flow in our usual channel, while the open circles represent flow with an initial narrow finger.

where the perturbation could be considered ''microscopic.'' In the range $l_c \approx d$, flows at both 0° and 45° show a return to the smooth ST curve, while flow at 30° narrow somewhat, but remain unusually wide. The limits of stability were approximately the same for all orientations, with the fingers stable up to about $1/B \approx 3000$. These fingers were in general much less stable than corresponding isotropic ST fingers in a similar cell, the latter being stable up to $1/B \approx 8000$.

We have also tried to force a narrow finger, to check for the possibility that our wide fingers represented long-lived transients. We used a channel which had an initial width of 2 cm and a length of several centimeters which opened up into a channel of a width of 3 cm. In this way we could set up an initial finger narrower than that expected in our experimental setup for a width of 3 cm, and investigate how (or if) the finger relaxed into a different steady state once past the narrow portion of the channel. If the wider fingers were long-lived transients, one might expect that the prepared narrow fingers might stay in a narrow state, if that were the true solution. While the fingers created using this initial condition were marginally ($\leqslant 1$ standard deviation) narrower than those observed starting from a flat interface, they were still very significantly wider than the isotropic ST curve. A comparison of the curves found with the normal channel and the narrowed entrance channel for angles of 0° and 45° are shown in Figs. 2 and 3. The fact that the fingers return to a wider solution even after being forced into a narrow initial solution suggests that the wider solution was not a long-lived transient.

IV. DISCUSSION OF RESULTS

We can suggest a tentative approach to thinking about the results presented above. Sarkar and Jasnow [38] have argued that if the capillary length is much greater than the lattice cell size, then it may be legitimate to use a coarse graining argument to represent an otherwise very complicated mobility tensor in Darcy's law

$$\vec{v} = -\underline{\mathbf{M}}\nabla\tilde{\mathbf{P}}, \tag{3}$$

where $\vec{v}$ is the observed interface velocity, $\underline{\mathbf{M}}$ is the coarse grained mobility tensor, and $\tilde{\mathbf{P}}$ is the pressure field. If coarse graining is valid, this mobility tensor should have equal eigenvalues for a lattice with fourfold or sixfold symmetry. In this case, the equation of flow for our square lattice would remain a Laplace equation. One must also consider possible changes in the boundary conditions due to the perturbation. For a smooth cell (no lattice at all), the boundary conditions are that the normal components of velocity for the two fluids must match

$$(v_n)_1 = (v_n)_2 \tag{4}$$

and that the pressure jumps across the two fluids, one of which was assumed to wet the plates, should have the form

$$\Delta P = \frac{T}{b/2}\left[1+3.80\left(\frac{\mu v_n}{T}\right)^{2/3} + \text{higher order terms in } v\right] + \frac{\pi}{4}T\kappa, \tag{5}$$

where κ is the curvature in the plane of the channel. The presence of the lattice can be expected to change these boundary conditions, and questions connected with hydrodynamic thinning of the oil wetting the etched glass surface make it not at all clear how to write down the corrected boundary condition, even though these should reflect the anisotropy. One might expect the anisotropy to bring in both an angular dependence to the surface tension T in Eq. (5) and also a kinetic (v_n dependent) term. Although there is a significant amount of theoretical work which includes anisotropic terms in the surface tension, possible kinetic anisotropy remains largely unexplored for ST fingering in the channel geometry.

In the only work modeling both kinds of possible anisotropy in the channel geometry, ST fingering with both surface tension and kinetic anisotropy has been modeled by Li, Kessler, and Sander [27], although the emphasis was on the possibility of sidebranching in the steady-state finger under these conditions with the addition of periodic or white noise. The surface tension anisotropy was modeled with a term in the surface tension of the form $1-[\epsilon\cos(4\theta)]$ (here they assume fourfold symmetry for the anisotropy). The kinetic term was modeled with a term in the pressure jump of the form $-u_n(x)g(\theta)$, where $g(\theta)=\beta[1-\cos(4\theta)]$ and β is a small constant. For ϵ positive and $\beta=0$, narrow fingers were observed for some values of ϵ, while for larger values ($\epsilon>0.4$) fingers wider and more unstable than the unperturbed fingers were seen. These results were not discussed in detail in their report. With both kinetic and positive surface tension anisotropy included, no sidebranches were observed without the addition of either white or periodic noise. Sidebranches were observed without noise when ϵ was negative, corresponding to a maximum in the surface tension term.

Dorsey and Martin [25] have considered the effect of putting just an anisotropic surface tension into the equations for flow in the channel geometry. They investigated a surface tension term $T(\theta)$ which had the form $T(\theta)=T[1-2\delta\cos(m\theta)]$ where δ was a positive constant, and θ was the angle between the normal **n** to the interface and the channel axis. Since δ was assumed positive, this leads to a surface tension minimum at the tip of the finger. They showed that this anisotropic surface tension term allowed fingers of all widths, and may lead to fingers narrower than the normal ST solution. Ben Amar, Combescot, and Couder [23] and Combescot [39] have done a similar analysis for the case of a similar surface tension maximum at the fingertip, using an anisotropic surface tension of the form: $T(\theta)=T[1+2\delta\cos(M\theta)]$ where δ was again a positive constant and θ was the angle between the normal **n** and the channel axis. Ben Amar, Combescot, and Couder first discovered a possible ''exceptional solution'' in the case of a surface tension maximum. This exceptional solution has wider fingers than the smooth ST case, and the fingers widen with increasing velocity. Combescot [39] has extended the study of this exceptional solution at a surface tension maximum in the small surface tension (large $1/B$) limit, confirming the calculated results of Ben Amar, Combescot, and Couder [23] and arguing that the exceptional solution should become wider as $1/B$ increases.

Couder *et al.* [21] and Ben Amar, Combescot, and Couder [23] reported results from experiments involving high-velocity, unstable fingering over a nylon mesh in a Hele-Shaw cell. The direction of easy growth was assumed to be along the threads of the mesh. Couder *et al.* [21] looked briefly at flows oriented at both 0° and 45° to the threads, and noted the resemblance of the unstable and highly branched fingers to DLA-like structures. Ben Amar, Combescot, and Couder [23] have done a more thorough study of flow oriented at 45° to the direction of the threads, which they argued should be the direction in which the surface tension will be a maximum, since they assume that the direction of easy growth will be along the threads. Both groups worked in a high-velocity regime, and studied unstable, DLA-like fingers. After averaging over many flow realizations, they constructed average finger patterns with widths above the ST result for flow at 45°, and below the ST result for flow at 0°. Since the preponderance of their individual flow patterns were very unstable and far into the DLA-like regime, it was unclear how to compare their results with ours, which were obtained from stable and reproducible flows. Further caution in interpreting the similarities of the two experiments in producing fatter fingers is warranted because: (1) for most of their data their capillary lengths were small compared to their lattice constants; (2) there may be differences due to the different wetting properties of nylon and glass, and (3) a stretched-nylon lattice may be less homogeneous and more deformable than the etched plate that we have used.

Corvera, Guo, and Jasnow [40], using the Hele-Shaw equations with an anisotropic surface tension and no kinetic term, have used both analytic and computational techniques to study finger width as a function of $1/B$ for two cases, one of which has surface tension T a local maximum at the ST fingertip and the other with T a local minimum at the fingertip. In the case where T is a local minimum at the fingertip, they find that the pattern evolves in time, but converges to a solution indistinguishable from the normal ST finger. When T is a local maximum, the normal ST finger is also indistinguishable from their selected solution below a threshold in $1/B$. Above this threshold, they observe selection of a finger wider than the ST finger.

In our experiment, we had hoped to be able to separate surface tension effects from kinetic effects by exploring a wide range of driving velocities. In this way, there was the possibility that any kinetic effects would be negligible at low velocities, allowing us to study the surface tension anisotropy alone. In analyzing our results, we have made no *a priori* assumptions concerning which direction of flow might correspond to a maximum or a minimum in the surface tension, although we suspected that 0° and 45° were likely candidates, due to the fourfold symmetry of the lattice. Wavelengths seen in flows at all measured lattice orientations fall on the same curve as those seen in pure ST flow, as long as $1/B$ values are small. Beyond a low velocity threshold, but still in the regime of $1/B$ where the capillary length is small compared to the lattice cell size ($1/B \approx 3000$), our data show that 45° flows are closer to the isotropic ST curve than those at either 0° or 30°. If we try to interpret these data as involving purely surface tension anisotropy as modeled by Corvera, Guo, and Jasnow [40], then it should not be possible to have fingers wider than the ST solution at all angles from 0° to 45°. At no orientations do we see fingers as narrow as the isotropic ST case. We have checked for the possibility of the wide fingers at 0° and 45° being long-lived transients, as discussed above, and have convinced ourselves that long-lived transients are not a likely explanation of our observations.

While we do see some qualitative agreement with the theoretical work of Corvera, Guo, and Jasnow, there are clearly effects present in the experiment which have not been addressed in the theory. A likely candidate is the absence of kinetic effects in the theory. It may be that kinetic effects tend to increase the width of the fingers, and that was why we see wide fingers at all flow orientations when we expected to see fingers lying on the isotropic ST curve in the case of a surface tension minimum. If that is the case, then possibly the extra width of the 0° fingers may indicate that the surface tension maximum of Corvera, Guo, and Jasnow [40] is present at $\theta=0°$.

It is unclear how to interpret our results at large values of $1/B$. In that case, the perturbation should no longer be microscopic and the mobility tensor in Darcy's law could well be some complicated function of position, since the coarse-graining argument used by Sarkar and Jasnow would no longer apply.

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